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Second-Order Multi-Agent Consensus via Dynamic Displacement Interaction

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Abstract: This note deals with a second-order multi-agent consensus via a dynamic displacement interaction, where it is assumed that velocity of each agent is not available. It is shown that there exists a dynamic displacement interaction which achieves the consensus if the graph of the overall system is undirected and connected. This sufficient condition for the consensus is presented by using graph Laplacians.

Keywords: multi-agent system, consensus, second-order differential equation, dynamic displacement interaction.

1. INTRODUCTION

Multi-agent consensus theory has been established well nowadays for standard first-order agents [1, 2], while the second-order agent case is still under investigation. In fact, when the agent is represented as the double integrator, it is impossible to achieve consensus via an interaction with static weights if velocity of each agent is not measurable. Thus, in the existing results [3-5], velocity as well as displacement of each agent are assumed to be available, though it is indeed a strong assumption for a certain practical case.

In this note, we establish a consensus theory for a second-order (double integrator) multi-agent system, where velocity of each agent is not assumed to be measurable. Instead of static weights, we employ dynamic weights in agent interaction. Then we show that such a dynamic interaction achieves the consensus if the graph of the overall system is undirected and connected. We also give a sufficient condition for the consensus, where it is represented by using graph Laplacians.

2. SECOND-ORDER MULTI-AGENT CONSENSUS

Let us start at a basic result about second-order multi-agent consensus. We consider N double integrator agents

$$\ddot{x}_i(t) = u_i(t), \qquad i = 1, 2, \ldots, N \tag{1}$$

and *two kinds of interactions*

$$u_i(t) = -\sum_{j=1}^{N} \{p_{ij}(\dot{x}_i(t) - \dot{x}_j(t)) + q_{ij}(x_i(t) - x_j(t))\},$$
$$i = 1, 2, \ldots, N \tag{2}$$

with *static weights* $p_{ij} \geq 0$ and $q_{ij} \geq 0$, where $x_i(t) \in \mathbb{R}$, $\dot{x}_i(t) \in \mathbb{R}$, and $\ddot{x}_i(t) \in \mathbb{R}$ are the displacement, the velocity, and the acceleration of agent i and $u_i(t) \in \mathbb{R}$ is the input to agent i.

When we define the displacement vector as

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}^T,$$

we have a representation of the overall system

$$\ddot{x}(t) + L_v \dot{x}(t) + L_d x(t) = 0, \tag{3}$$

where L_v and L_d are the graph Laplacians whose (i, j) off-diagonal elements are $-p_{ij}$ and $-q_{ij}$ and whose (i, i) diagonal elements are $\sum_{j=1}^{N} p_{ij}$ and $\sum_{j=1}^{N} q_{ij}$, respectively. It is well known that, if the graphs corresponding to L_v and L_d are undirected and connected, these L_v and L_d satisfy

$$L_v = L_v^T \geq 0, \qquad L_v \mathbf{1}_N = 0, \qquad \operatorname{rank} L_v = N - 1, \tag{4}$$
$$L_d = L_d^T \geq 0, \qquad L_d \mathbf{1}_N = 0, \qquad \operatorname{rank} L_d = N - 1, \tag{5}$$

where $\mathbf{1}_N \in \mathbb{R}^N$ is the vector whose elements are all one.

Then we have the following fact.

Theorem 1: The system (3) achieves a consensus

$$\lim_{t\to\infty} (\dot{x}(t) - a\mathbf{1}_N) = 0, \tag{6}$$
$$\lim_{t\to\infty} (x(t) - (at + b)\mathbf{1}_N) = 0 \tag{7}$$

if L_v and L_d satisfy (4) and (5), where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are some constants.

To see this fact, let us define a matrix $S \in \mathbb{R}^{N\times(N-1)}$ which satisfies

$$\begin{bmatrix} \mathbf{1}_N^T / \sqrt{N} \\ S^T \end{bmatrix} \begin{bmatrix} \mathbf{1}_N / \sqrt{N} & S \end{bmatrix} = \begin{bmatrix} \mathbf{1}_N / \sqrt{N} & S \end{bmatrix} \begin{bmatrix} \mathbf{1}_N^T / \sqrt{N} \\ S^T \end{bmatrix} = I_N,$$

where $I_N \in \mathbb{R}^{N\times N}$ is the identity matrix. It should be noted that such an orthonormal complement S to $\mathbf{1}_N / \sqrt{N}$ always exists. With this S, we introduce a variable transformation and its inverse as

$$\begin{bmatrix} \bar{x}(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_N^T / \sqrt{N} \\ S^T \end{bmatrix} x(t), \tag{8}$$

$$x(t) = \begin{bmatrix} \mathbf{1}_N / \sqrt{N} & S \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \tilde{x}(t) \end{bmatrix}. \tag{9}$$

Then we see that the overall system (3) can be represented as a set of differential equations

$$\ddot{\bar{x}}(t) = 0, \tag{10}$$

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$$\ddot{\tilde{x}}(t) + S^T L_v S\,\dot{\tilde{x}}(t) + S^T L_d S\,\tilde{x}(t) = 0. \tag{11}$$

Since the properties (4) and (5) of the graph Laplacian of undirected and connected graphs guarantee $S^T L_v S > 0$ and $S^T L_d S > 0$, we have

$$\lim_{t\to\infty} \dot{\tilde{x}}(t) = 0, \qquad \lim_{t\to\infty} \tilde{x}(t) = 0.$$

This fact follows a stability condition [6] of a second-order differential equation for the system (11). Employing this fact together with (8) and (9), we have

$$\begin{aligned}
\lim_{t\to\infty}\left(\dot{x}(t) - \mathbf{1}_N \frac{\mathbf{1}_N^T \dot{x}(t)}{N}\right) &= \lim_{t\to\infty}\left(\dot{x}(t) - \frac{\mathbf{1}_N}{\sqrt{N}}\dot{\bar{x}}(t)\right)\\
&= S \lim_{t\to\infty} \dot{\tilde{x}}(t) = 0,\\
\lim_{t\to\infty}\left(x(t) - \mathbf{1}_N \frac{\mathbf{1}_N^T x(t)}{N}\right) &= \lim_{t\to\infty}\left(x(t) - \frac{\mathbf{1}_N}{\sqrt{N}}\bar{x}(t)\right)\\
&= S \lim_{t\to\infty} \tilde{x}(t) = 0,
\end{aligned}$$

which means that a consensus is achieved for $\dot{x}(t)$ and $x(t)$. The consensus values are given as

$$\begin{aligned}
\frac{\mathbf{1}_N^T \dot{x}(t)}{N} &= \frac{\dot{\bar{x}}(t)}{\sqrt{N}} = \frac{\dot{\bar{x}}(0)}{\sqrt{N}} = \frac{\mathbf{1}_N^T \dot{x}(0)}{N} = \frac{1}{N}\sum_{j=1}^{N} \dot{x}_i(0) = a,\\
\frac{\mathbf{1}_N^T x(t)}{N} &= \frac{1}{N}\sum_{j=1}^{N} x_i(t) = at + b
\end{aligned}$$

for some constants a and b since $\dot{\bar{x}}(t)$ is a constant from (10) and thus $\dot{\bar{x}}(t) = \dot{\bar{x}}(0)$. In this way, **Theorem 1** can be established.

Notice here that the above discussion is based on availability of velocity $\dot{x}_i(t)$, which is in fact a strong assumption for a certain practical case. However, if $\dot{x}_i(t)$ is not measurable, the control

$$u_i(t) = -\sum_{j=1}^{N} q_{ij}(x_i(t) - x_j(t)), \qquad i = 1, 2, \dots, N,$$

with *static weights* q_{ij} cannot achieve the consensus. In fact, in this case, we have the overall system

$$\ddot{x}(t) + L_d x(t) = 0,$$

which says that all of the modes are on the imaginary axis. That is, the consensus cannot be not achieved.

The purpose of this note is to present a consensus theory for second-order multi-agent systems with *dynamic displacement interaction*. To this end, we employ *a stability theory for second-order differential equations* [7]. We here remark that the transfer function of the second-order agent from $u_i(t)$ to $x_i(t)$ is $1/s^2$, and thus it is *not passive*. Although passivity plays a crucial role in standard multi-agent consensus [8], it cannot help us if we consider the second-order agent without the velocity measurement $\dot{x}_i(t)$. This is the reason why we need a different way for our analysis.

3. DYNAMIC DISPLACEMENT INTERACTION

In this note, we consider the following type of interaction with dynamic weights

$$\begin{aligned}
\dot{z}_i(t) &= -\gamma z_i(t) - \gamma \sum_{j=1}^{N} (\gamma p_{ij} - q_{ij})(x_i(t) - x_j(t)),\\
u_i(t) &= -z_i(t) - \gamma \sum_{j=1}^{N} p_{ij}(x_i(t) - x_j(t)),\\
&\quad i = 1, 2, \dots, N
\end{aligned} \tag{12}$$

with $p_{ij} \geq 0$ and $q_{ij} \geq 0$, where $z_i(t) \in \mathbb{R}$ is the state of the dynamic weight. The parameter $\gamma > 0$ of the weights will be determined later.

This dynamic interaction can be represented as

$$u_i(s) = -\sum_{j=1}^{N} w_{ij}(s)(x_i(s) - x_j(s)), \qquad i = 1, 2, \dots, N$$

in frequency domain, where the transfer functions of the dynamic weights are

$$w_{ij}(s) = \gamma p_{ij} - \frac{\gamma(\gamma p_{ij} - q_{ij})}{s + \gamma} = \frac{p_{ij}s + q_{ij}}{(1/\gamma)s + 1}.$$

That is, if γ tends to infinity, $w_{ij}(s)$ tends to $p_{ij}s + q_{ij}$, which implies that (12) also tends to (2). In this sense, this dynamic interaction will achieve the consensus with a suitable choice of γ. Then, the question is *how large we have to make* γ in order to achieve the consensus.

When we define the state of the dynamic weights as

$$z(t) = \begin{bmatrix} z_1(t) & z_2(t) & \cdots & z_N(t) \end{bmatrix}^T,$$

we have a representation of the overall system

$$\begin{aligned}
&\ddot{x}(t) + \gamma L_v x(t) + z(t) = 0,\\
&\frac{1}{\gamma}\dot{z}(t) + (\gamma L_v - L_d)x(t) + z(t) = 0,
\end{aligned} \tag{13}$$

where L_v and L_d are the graph Laplacians defined in the same way for (3). Thus, if the graphs corresponding to L_v and L_d are undirected and connected, these L_v and L_d also satisfy (4) and (5).

Here we have the main result of this note.

Theorem 2: The system (13) achieves a consensus

$$\lim_{t\to\infty} (\dot{x}(t) - a\mathbf{1}_N) = 0, \tag{14}$$

$$\lim_{t\to\infty} (x(t) - (at + b)\mathbf{1}_N) = 0, \tag{15}$$

$$\lim_{t\to\infty} z(t) = 0 \tag{16}$$

if L_v and L_d satisfy (4) and (5) and γ satisfies

$$S^T(\gamma L_v - L_d)S > 0, \tag{17}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are some constants.

The proof of this theorem is as follows. We first introduce a variable transformation

$$z(t) = \left(\gamma L_v - L_d + \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}\right)\hat{z}(t), \tag{18}$$

where we see that $\gamma L_v - L_d + \mathbf{1}_N \mathbf{1}_N^T / N > 0$ from (17) and thus this is a nonsingular transformation. Then we have another representation of the overall system

$$\begin{aligned}
&\ddot{x}(t) + \gamma L_v x(t) + \left(\gamma L_v - L_d + \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}\right)\hat{z}(t) = 0,\\
&\frac{1}{\gamma}\left(\gamma L_v - L_d + \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}\right)\dot{\hat{z}}(t) + (\gamma L_v - L_d)x(t)\\
&\qquad\qquad + \left(\gamma L_v - L_d + \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}\right)\hat{z}(t) = 0.
\end{aligned}$$

We further employ a variable transformation for $\hat{z}(t)$

$$\begin{bmatrix} \bar{z}(t) \\ \tilde{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_N^T / \sqrt{N} \\ S^T \end{bmatrix} \hat{z}(t), \tag{19}$$

$$\hat{z}(t) = \begin{bmatrix} \mathbf{1}_N / \sqrt{N} & S \end{bmatrix} \begin{bmatrix} \bar{z}(t) \\ \tilde{z}(t) \end{bmatrix} \tag{20}$$

as well as the variable transformation (8) and (9) for $x(t)$. Then we can describe the overall system as

$$\ddot{\bar{x}}(t) + \bar{z}(t) = 0, \tag{21}$$

$$\ddot{\tilde{x}}(t) + S^T \gamma L_v S\, \tilde{x}(t) + S^T (\gamma L_v - L_d) S\, \tilde{z}(t) = 0, \tag{22}$$

$$\frac{1}{\gamma}\dot{\bar{z}}(t) + \bar{z}(t) = 0, \tag{23}$$

$$\begin{aligned}
&\frac{1}{\gamma} S^T (\gamma L_v - L_d) S\, \dot{\tilde{z}}(t) + S^T (\gamma L_v - L_d) S\, \tilde{x}(t)\\
&\qquad\qquad + S^T (\gamma L_v - L_d) S\, \tilde{z}(t) = 0,
\end{aligned} \tag{24}$$

where the coefficients of (22) and (24) satisfy

$$\begin{aligned}
&\frac{1}{\gamma} S^T (\gamma L_v - L_d) S > 0,\\
&\begin{bmatrix} \gamma S^T L_v S & S^T (\gamma L_v - L_d) S \\ S^T (\gamma L_v - L_d) S & S^T (\gamma L_v - L_d) S \end{bmatrix} > 0,\\
&\operatorname{rank} S^T (\gamma L_v - L_d) S = N - 1
\end{aligned}$$

under the condition (17). Thus we see that

$$\lim_{t\to\infty} \dot{\tilde{x}}(t) = 0, \qquad \lim_{t\to\infty} \tilde{x}(t) = 0, \qquad \lim_{t\to\infty} \tilde{z}(t) = 0, \tag{25}$$

which follows a stability condition [7] for a dynamical system like (22) and (24) which is described by coupled differential equations of second order and first order. Since (23) says that

$$\lim_{t\to\infty} \bar{z}(t) = 0,$$

we have (16) with (18) and (20). Regarding $\dot{x}(t)$ and $x(t)$, using (25) with (8) and (9), we can obtain

$$\lim_{t\to\infty}\left(\dot{x}(t) - \mathbf{1}_N \frac{\mathbf{1}_N^T \dot{x}(t)}{N}\right) = S \lim_{t\to\infty} \dot{\tilde{x}}(t) = 0,$$

$$\lim_{t\to\infty}\left(x(t) - \mathbf{1}_N \frac{\mathbf{1}_N^T x(t)}{N}\right) = S \lim_{t\to\infty} \tilde{x}(t) = 0$$

as is done in the previous section. Here the consensus values are represented as

$$\frac{\mathbf{1}_N^T \dot{x}(t)}{N} = \frac{\dot{\bar{x}}(t)}{\sqrt{N}}, \qquad \frac{\mathbf{1}_N^T x(t)}{N} = \frac{\bar{x}(t)}{\sqrt{N}},$$

where we see that

$$\lim_{t\to\infty}(\dot{\bar{x}}(t) - a) = 0, \qquad \lim_{t\to\infty}(\bar{x}(t) - (at + b)) = 0$$

for some constants a and b since

$$\lim_{t\to\infty} \ddot{\bar{x}}(t) = -\lim_{t\to\infty} \bar{z}(t) = 0$$

from (21). We therefore see that **Theorem 2** holds true.

4. CONCLUDING REMARKS

In this note, we have investigated a second-order multi-agent consensus via a dynamic displacement interaction, where it has been assumed that velocity of each agent is not measurable. We have established a sufficient condition for the consensus, where the condition is represented by using graph Laplacians. We have seen that such a dynamic interaction achieves the consensus always exists if the graph of the overall system is undirected and connected, where the dynamics of the weights of the interaction should be selected adequately.

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